Spring 2012

EE 330

ENGINEERING ELECTROMAGNETICS

HW 8: *Due Friday 16 March*

5.22, 5.40, 5.41, 6.7, 6.12, 6.16, 6.19, 6.23

Problem 5.22 A long cylindrical conductor whose axis is coincident with the z-axis has a radius a and carries a current characterized by a current density $\mathbf{J} = \hat{\mathbf{z}}J_0/r$, where J_0 is a constant and r is the radial distance from the cylinder's axis. Obtain an expression for the magnetic field $\mathbf{H}$ for

(a) $0 \le r \le a$

(b) $r > a$

Solution: This problem is very similar to Example 5-5.

(a) For $0 \le r_1 \le a$, the total current flowing within the contour C_1 is

$$I_1 = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{r_1} \left(\frac{\hat{\mathbf{z}}J_0}{r} \right) \cdot (\hat{\mathbf{z}} r \, dr \, d\phi) = 2\pi \int_{r=0}^{r_1} J_0 \, dr = 2\pi r_1 J_0.$$

Therefore, since $I_1 = 2\pi r_1 H_1$, $H_1 = J_0$ within the wire and $\mathbf{H}_1 = \hat{\boldsymbol{\phi}} J_0$.

(b) For $r \ge a$, the total current flowing within the contour is the total current flowing within the wire:

$$I = \iint \mathbf{J} \cdot d\mathbf{s} = \int_{\phi=0}^{2\pi} \int_{r=0}^{a} \left(\frac{\hat{\mathbf{z}}J_0}{r} \right) \cdot (\hat{\mathbf{z}} r \, dr \, d\phi) = 2\pi \int_{r=0}^{a} J_0 \, dr = 2\pi a J_0.$$

Therefore, since $I = 2\pi r H_2$, $H_2 = J_0 a/r$ within the wire and $\mathbf{H}_2 = \hat{\boldsymbol{\phi}} J_0 (a/r)$.

Problem 5.40 The rectangular loop shown in Fig. P5.40 is coplanar with the long, straight wire carrying the current $I = 20$ A. Determine the magnetic flux through the loop.

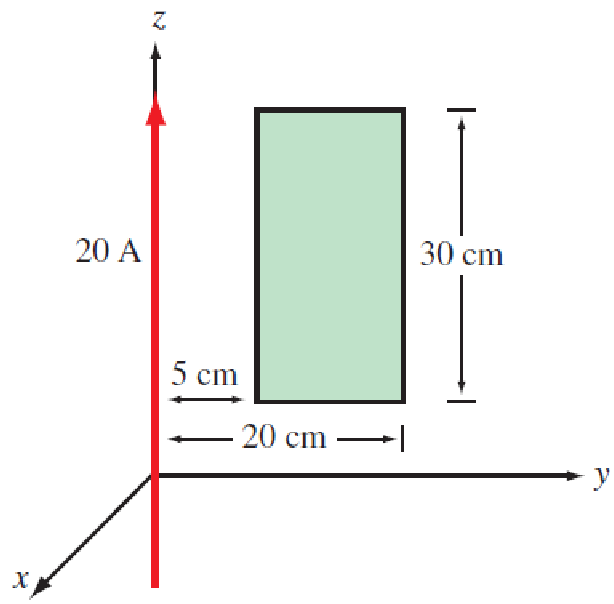

Figure P5.40: Loop and wire arrangement for Problem 5.40.

Solution: The field due to the long wire is, from Eq. (5.30),

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi r} = -\hat{\mathbf{x}} \frac{\mu_0 I}{2\pi y},$$

where in the plane of the loop, $\hat{\boldsymbol{\phi}}$ becomes $-\hat{\mathbf{x}}$ and r becomes y.

The flux through the loop is along $-\hat{\mathbf{x}}$, and the magnitude of the flux is

$$\begin{aligned}
\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} &= \frac{\mu_0 I}{2\pi} \int_{5 \text{ cm}}^{20 \text{ cm}} -\frac{\hat{\mathbf{x}}}{y} \cdot -\hat{\mathbf{x}} \, (30 \text{ cm} \times dy) \\
&= \frac{\mu_0 I}{2\pi} \times 0.3 \int_{0.05}^{0.2} \frac{dy}{y} \\
&= \frac{0.3\,\mu_0}{2\pi} \times 20 \times \ln\left(\frac{0.2}{0.05}\right) = 1.66 \times 10^{-6} \quad \text{(Wb)}.
\end{aligned}$$

Problem 5.41 Determine the mutual inductance between the circular loop and the linear current shown in Fig. P5.41.

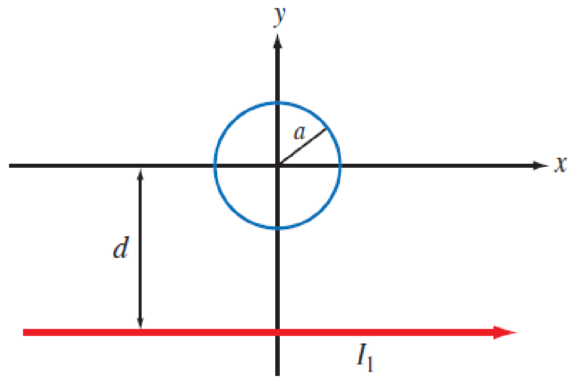

Figure P5.41: Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 5.41).

Solution: To calculate the magnetic flux through the loop due to the current in the conductor, we consider a thin strip of thickness dy at location y, as shown. The magnetic field is the same at all points across the strip because they are all equidistant (at $r = d + y$) from the linear conductor. The magnetic flux through the strip is

$$\begin{aligned}
d\Phi_{12} &= \mathbf{B}(y) \cdot d\mathbf{s} = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi(d+y)} \cdot \hat{\mathbf{z}} 2(a^2 - y^2)^{1/2}\, dy \\
&= \frac{\mu_0 I (a^2 - y^2)^{1/2}}{\pi(d+y)}\, dy \\
L_{12} &= \frac{1}{I} \int_S d\Phi_{12} \\
&= \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{(a^2 - y^2)^{1/2}\, dy}{(d+y)}
\end{aligned}$$

Let $z = d + y \rightarrow dz = dy$. Hence,

$$\begin{aligned}
L_{12} &= \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{a^2 - (z-d)^2}}{z}\, dz \\
&= \frac{\mu_0}{\pi} \int_{d-a}^{d+a} \frac{\sqrt{(a^2 - d^2) + 2dz - z^2}}{z}\, dz \\
&= \frac{\mu_0}{\pi} \int \frac{\sqrt{R}}{z}\, dz
\end{aligned}$$

where $R = a_0 + b_0 z + c_0 z^2$ and

$$\begin{aligned}
a_0 &= a^2 - d^2 \\
b_0 &= 2d \\
c_0 &= -1 \\
\Delta &= 4a_0 c_0 - b_0^2 = -4a^2 < 0
\end{aligned}$$

From Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1980, p. 84), we have

$$\int \frac{\sqrt{R}}{z}\, dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}\,.$$

For

$$\sqrt{R}\Big|_{z=d-a}^{d+a} = \sqrt{a^2 - d^2 + 2dz - z^2}\Big|_{z=d-a}^{d+a} = 0 - 0 = 0.$$

For $\int \frac{dz}{z\sqrt{R}}$, several solutions exist depending on the sign of a_0 and Δ.

For this problem, $\Delta < 0$, also let $a_0 < 0$ (i.e., $d > a$). Using the table of integrals,

$$\begin{aligned}
a_0 \int \frac{dz}{z\sqrt{R}} &= a_0 \left[\frac{1}{\sqrt{-a_0}} \sin^{-1}\left(\frac{2a_0 + b_0 z}{z\sqrt{b_0^2 - 4a_0 c_0}} \right) \right]_{z=d-a}^{d+a} \\
&= -\sqrt{d^2 - a^2} \left[\sin^{-1}\left(\frac{a^2 - d^2 + dz}{az} \right) \right]_{z=d-a}^{d+a} \\
&= -\pi\sqrt{d^2 - a^2}\,.
\end{aligned}$$

For $\int \frac{dz}{\sqrt{R}}$, different solutions exist depending on the sign of c_0 and Δ.

In this problem, $\Delta < 0$ and $c_0 < 0$. From the table of integrals,

$$\begin{aligned}
\frac{b_0}{z} \int \frac{dz}{\sqrt{R}} &= \frac{b_0}{2} \left[\frac{-1}{\sqrt{-c_0}} \sin^{-1} \frac{2c_0 z + b_0}{\sqrt{-\Delta}} \right]_{z=d-a}^{d+a} \\
&= -d \left[\sin^{-1}\left(\frac{d - z}{a} \right) \right]_{z=d-a}^{d+a} = \pi d.
\end{aligned}$$

Thus

$$\begin{aligned}
L_{12} &= \frac{\mu_0}{\pi} \cdot \left[\pi d - \pi\sqrt{d^2 - a^2} \right] \\
&= \mu_0 \left[d - \sqrt{d^2 - a^2} \right].
\end{aligned}$$

Problem 6.7 The rectangular conducting loop shown in Fig. P6.7 rotates at 6,000 revolutions per minute in a uniform magnetic flux density given by

$$\mathbf{B} = \hat{\mathbf{y}}\, 50 \quad (\text{mT}).$$

Determine the current induced in the loop if its internal resistance is 0.5 Ω.

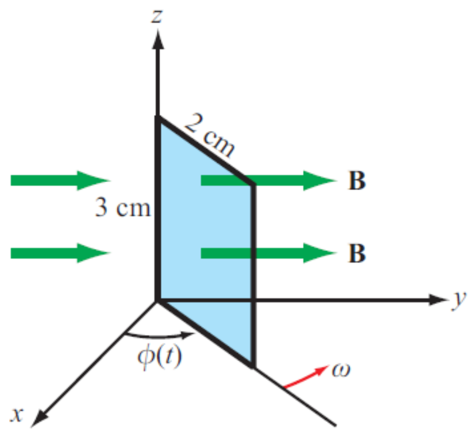

Figure P6.7: Rotating loop in a magnetic field (Problem 6.7).

Solution:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \hat{\mathbf{y}}\, 50 \times 10^{-3} \cdot \hat{\mathbf{y}}(2 \times 3 \times 10^{-4}) \cos\phi(t) = 3 \times 10^{-5} \cos\phi(t),$$

$$\phi(t) = \omega t = \frac{2\pi \times 6 \times 10^3}{60} t = 200\pi t \quad (\text{rad/s}),$$

$$\Phi = 3 \times 10^{-5} \cos(200\pi t) \quad (\text{Wb}),$$

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = 3 \times 10^{-5} \times 200\pi \sin(200\pi t) = 18.85 \times 10^{-3} \sin(200\pi t) \quad (\text{V}),$$

$$I_{\text{ind}} = \frac{V_{\text{emf}}}{0.5} = 37.7 \sin(200\pi t) \quad (\text{mA}).$$

The direction of the current is CW (if looking at it along $-\hat{\mathbf{x}}$-direction) when the loop is in the first quadrant ($0 \leq \phi \leq \pi/2$). The current reverses direction in the second quadrant, and reverses again every quadrant.

Problem 6.12 The electromagnetic generator shown in Fig. 6-12 is connected to an electric bulb with a resistance of 150 Ω. If the loop area is 0.1 m^2 and it rotates at 3,600 revolutions per minute in a uniform magnetic flux density $B_0 = 0.4$ T, determine the amplitude of the current generated in the light bulb.

Solution: From Eq. (6.38), the sinusoidal voltage generated by the a-c generator is $V_{\text{emf}} = A\omega B_0 \sin(\omega t + C_0) = V_0 \sin(\omega t + C_0)$. Hence,

$$V_0 = A\omega B_0 = 0.1 \times \frac{2\pi \times 3{,}600}{60} \times 0.4 = 15.08 \quad (\text{V}),$$

$$I = \frac{V_0}{R} = \frac{15.08}{150} = 0.1 \quad (\text{A}).$$

Problem 6.16 The parallel-plate capacitor shown in Fig. P6.16 is filled with a lossy dielectric material of relative permittivity ε_r and conductivity σ. The separation between the plates is d and each plate is of area A. The capacitor is connected to a time-varying voltage source $V(t)$.

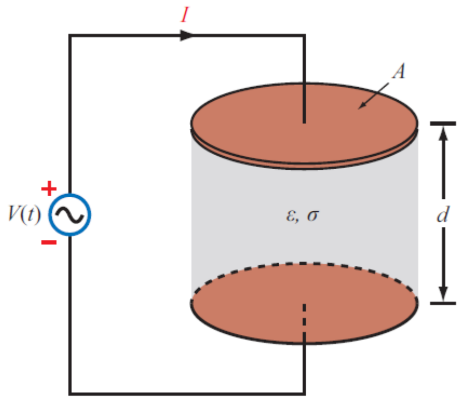

Figure P6.16: Parallel-plate capacitor containing a lossy dielectric material (Problem 6.16).

(a) Obtain an expression for I_c, the conduction current flowing between the plates inside the capacitor, in terms of the given quantities.

(b) Obtain an expression for I_d, the displacement current flowing inside the capacitor.

(c) Based on your expressions for parts (a) and (b), give an equivalent-circuit representation for the capacitor.

(d) Evaluate the values of the circuit elements for $A = 4$ cm^2, $d = 0.5$ cm, $\varepsilon_r = 4$, $\sigma = 2.5$ (S/m), and $V(t) = 10\cos(3\pi \times 10^3 t)$ (V).

Solution:

(a)

$$R = \frac{d}{\sigma A}, \qquad I_c = \frac{V}{R} = \frac{V\sigma A}{d}.$$

(b)

$$E = \frac{V}{d}, \qquad I_d = \frac{\partial D}{\partial t} \cdot A = \varepsilon A \frac{\partial E}{\partial t} = \frac{\varepsilon A}{d} \frac{\partial V}{\partial t}.$$

(c) The conduction current is directly proportional to V, as characteristic of a resistor, whereas the displacement current varies as $\partial V/\partial t$, which is characteristic of a capacitor. Hence,

$$R = \frac{d}{\sigma A} \quad \text{and} \quad C = \frac{\varepsilon A}{d}.$$

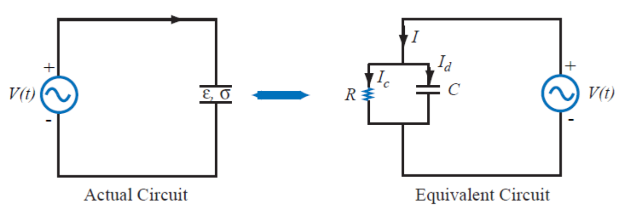

Figure P6.16: (a) Equivalent circuit.

(d)

$$R = \frac{0.5 \times 10^{-2}}{2.5 \times 4 \times 10^{-4}} = 5\ \Omega,$$

$$C = \frac{4 \times 8.85 \times 10^{-12} \times 4 \times 10^{-4}}{0.5 \times 10^{-2}} = 2.84 \times 10^{-12}\ \text{F}.$$

Problem 6.19 At $t = 0$, charge density ρ_{v0} was introduced into the interior of a material with a relative permittivity $\varepsilon_r = 9$. If at $t = 1$ μs the charge density has dissipated down to $10^{-3}\rho_{v0}$, what is the conductivity of the material?

Solution: We start by using Eq. (6.61) to find τ_r:

$$\rho_v(t) = \rho_{v0}e^{-t/\tau_r},$$

or

$$10^{-3}\rho_{v0} = \rho_{v0}e^{-10^{-6}/\tau_r},$$

which gives

$$\ln 10^{-3} = -\frac{10^{-6}}{\tau_r},$$

or

$$\tau_r = -\frac{10^{-6}}{\ln 10^{-3}} = 1.45 \times 10^{-7} \quad \text{(s)}.$$

But $\tau_r = \varepsilon/\sigma = 9\varepsilon_0/\sigma$. Hence

$$\sigma = \frac{9\varepsilon_0}{\tau_r} = \frac{9 \times 8.854 \times 10^{-12}}{1.45 \times 10^{-7}} = 5.5 \times 10^{-4} \quad \text{(S/m)}.$$

Problem 6.23 The electric field of an electromagnetic wave propagating in air is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}4\cos(6 \times 10^8 t - 2z) + \hat{\mathbf{y}}3\sin(6 \times 10^8 t - 2z) \quad \text{(V/m)}.$$

Find the associated magnetic field $\mathbf{H}(z,t)$.

Solution: Converting to phasor form, the electric field is given by

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}4e^{-j2z} - j\hat{\mathbf{y}}3e^{-j2z} \quad \text{(V/m)},$$

which can be used with Eq. (6.87) to find the magnetic field:

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= \frac{1}{-j\omega\mu}\nabla\times\widetilde{\mathbf{E}} \\
&= \frac{1}{-j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4e^{-j2z} & -j3e^{-j2z} & 0 \end{vmatrix} \\
&= \frac{1}{-j\omega\mu}(\hat{\mathbf{x}}6e^{-j2z} - \hat{\mathbf{y}}j8e^{-j2z}) \\
&= \frac{j}{6 \times 10^8 \times 4\pi \times 10^{-7}}(\hat{\mathbf{x}}6 - \hat{\mathbf{y}}j8)e^{-j2z} = j\hat{\mathbf{x}}8.0e^{-j2z} + \hat{\mathbf{y}}10.6e^{-j2z} \quad \text{(mA/m)}.
\end{aligned}$$

Converting back to instantaneous values, this is

$$\mathbf{H}(t,z) = -\hat{\mathbf{x}}8.0\sin(6 \times 10^8 t - 2z) + \hat{\mathbf{y}}10.6\cos(6 \times 10^8 t - 2z) \quad \text{(mA/m)}.$$
